1. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a bilinear function.*

   (a) Prove that $\lim\limits_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$.

   (b) Using this or otherwise, prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$. Here $Df(a,b)$ denotes the total derivative of f at the point $(a,b) \in \mathbb{R}^{n+m}$.

Solution:

(a) Let $\{e_i\}_{i=1}^n$ and $\{g_j\}_{j=1}^m$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then for $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$,

$$h = \sum_{i=1}^n x_i e_i,\ \ k = \sum_{j=1}^m y_j g_j.$$

By bilinearity, we get

$$f(h,k) = \sum_{i=1}^n \sum_{j=1}^m x_i y_j f(e_i, g_j).$$

Also, $\|(h,k)\| = (\sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2)^{\frac{1}{2}}$. Hence for each i, $|x_i| \leq \|(h,k)\|$. Thus

$$\begin{aligned}
\frac{\|f(h,k)\|}{\|(h,k)\|} &= \frac{\|\sum\limits_{i=1}^n \sum\limits_{j=1}^m x_i y_j f(e_i,g_j)\|}{\|(h,k)\|} \\
&\leq \frac{\sum\limits_{i=1}^n \sum\limits_{j=1}^m |x_i||y_j|\|f(e_i,g_j)\|}{\|(h,k)\|} \\
&\leq \sum_{i=1}^n \sum_{j=1}^m |y_j|\|f(e_i,g_j)\| \\
&\to 0 \text{ as } y_j \to 0.
\end{aligned}$$

Hence $\lim\limits_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0$.

(b) The map $(x,y) \to f(a,y) + f(x,b)$ from $\mathbb{R}^{m+n} \to \mathbb{R}^p$ is linear because of the bilinearity of f. Now,

$$\begin{aligned}
\lim_{(p,q)\to(a,b)} \frac{\|f(p,q) - f(a,b) - f(a,q-b) - f(p-a,b)\|}{\|(p,q)-(a,b)\|} &= \lim_{(p,q)\to(a,b)} \frac{\|f(p,q) - f(a,q) - f(p-a,b)\|}{\|(p-a,q-b)\|} \\
&= \lim_{(p,q)\to(a,b)} \frac{\|f(p-a,q) - f(p-a,b)\|}{\|(p-a,q-b)\|} \\
&= \lim_{(p,q)\to(a,b)} \frac{\|f(p-a,q-b)\|}{\|(p-a,q-b)\|} \\
&= 0 \text{ (by } (a)).
\end{aligned}$$

Hence $Df(a,b)(x,y) = f(a,y) + f(x,b)$ gives the total derivative of f at the point $(a,b) \in \mathbb{R}^{n+m}$.

2. *(a) State the Inverse and the Implicit Function Theorems.*

   (b) Show that the system of equations $x = u^4 - u + uv + v^2, y = \cos(u) + \sin(v)$, can be solved for (u,v) as a continuously differentiable function F of (x,y), in some neighborhood of $(0,0)$, in such a way that $(u,v) = (0,0)$ when $(x,y) = (0,1)$. What is the differential of F at $(0,1)$?

   (c) Can the equation $xz + yz + \sin(x+y+z) = 0$ be solved, in a neighborhood of $(0,0,0)$ for z as a continuously differentiable function $z = g(x,y)$ of (x,y), with $g(0,0) = 0$?

Solution:

(a) i. **Theorem 0.1 (Inverse Function Theorem)** *[1] Suppose f is a continuously differentiable function of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n, f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then*

   A. there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, f is one-one on U and $f(U) = V$.

   B. if g is the inverse of f defined in V by $g(f(x)) = x$, $x \in U$, then g is continuously differentiable on V and $g(y) = (f'(g(y)))^{-1}$, $y \in V$.

   ii. **Theorem 0.2 (Implicit Function Theorem)** *[1] Let f be a continuously differentiable function of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$ such that $f(a,b) = 0$ for some point $(a,b) \in E$. Let $A = f'(a,b)$ and A_x and A_y be given by $A_x h = A(h,0)$ and $A_y(k) = A(0,k)$. Assume that A_x is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$ with $(a,b) \in U$ and $b \in W$ satisfying the following property:*
   For each $y \in W$, there exists a unique x such that

$$(x,y) \in U \text{ and } f(x,y) = 0.$$

   Define this x to be $g(y)$. Then g is a continuously differentiable mapping of W into $\mathbb{R}^n, g(b) = a, f(g(y),y) = 0$ and $g'(b) = -(A_x)^{-1}A_y$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f(u,v) = (u^4 - u + uv + v^2, \cos(u) + \sin(v))$. Then $f'(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $f'(0,0)$ is invertible and the inverse function theorem is applicable. $f(0,0) = (0,1)$. Hence there exist neighbourhoods U of $(0,0)$ and V of $(0,1)$ and a continuously differentiable function $F : V \to U$ such that $F(f(u,v)) = (u,v)$. In particular, $F(0,1) = (0,0)$. Further, $F'(x,y) = (f'(F(x,y)))^{-1}$, thus $F'(0,1) = (f'(F(0,1)))^{-1} = (f'(0,0))^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Let $f(z,(x,y)) = xz+yz+\sin(x+y+z)$. Then $f(0,(0,0)) = 0$ and $A = f'(0,(0,0)) = (1,1,1)$. Then $A_z = 1$ is invertible (as it is non-zero), thus the implicit function theorem is applicable. That is, there exist open sets $U \subset \mathbb{R}^3$, $W \subset \mathbb{R}^2$ with $(0,0,0) \in U, (0,0) \in W$ and a continuously differentiable mapping g on W such that $f(g(x,y),x,y) = 0$, $(x,y) \in W$. That is, z can be expressed as a continuously differentiable function $z = g(x,y)$ of (x,y), with $g(0,0) = 0$.

3. *(a) Give an example of a bounded real valued function on a rectangle in $\mathbb{R}^d$ (for any d) which is not integrable.*

   (b) Define a Jordan region, and also define a set with volume zero.

(c) *Prove that a bounded set $E \subset \mathbb{R}^d$ is a Jordan region, if and only if, the boundary of E has d-volume zero.*

Solution:

(a) Let R be a rectangle in $\mathbb{R}^d$. Take $f : R \to \mathbb{R}$ by $f = \begin{cases} 1 & x_i \in \mathbb{Q} \text{ for any } i, \\ 0 & , \text{ otherwise.} \end{cases}$

Then f is bounded but not integrable on $\mathbb{R}$. It can be seen that the upper sum of f given by $U(f, \mathcal{G}) = \sum_{R_j \in \mathcal{G}} M_j |R_j| = \sum_{R_j \in \mathcal{G}} |R_j|$ for every grid $\mathcal{G}$ on R, where $M_j = \sup_{x \in R_j} f(x) = 1 \ \forall j$. However for every grid $\mathcal{G}$, the lower sum of f, $L(f, \mathcal{G}) = \sum_{R_j \in \mathcal{G}} m_j |R_j| = 0$, where $m_j = \inf_{x \in R_j} f(x) = 0 \ \forall j$.

(b) [2] Let E be a bounded subset of $\mathbb{R}^d$. The outer sum of E with respect to a grid $\mathcal{G} = \{R_1, \cdots, R_p\}$ on a rectangle $R \subset \mathbb{R}^d$ is given by

$$V(E, \mathcal{G}) = \sum_{R_j \cap \bar{E}} |R_j|.$$

For a rectangle R containing E, the outer volume of E is defined as

$$\overline{\mathrm{Vol}}(E) = \inf_{\mathcal{G} \text{ grid on } R} V(E, \mathcal{G}),$$

and is independent of the rectangle R chosen. The inner sum of E with respect to a grid $\mathcal{G}$ on a rectangle R is given by

$$v(E, \mathcal{G}) = \sum_{R_j \subseteq E^\circ} |R_j|.$$

For a rectangle R containing E, the inner volume of E is defined as

$$\underline{\mathrm{Vol}}(E) = \sup_{\mathcal{G} \text{ grid on } R} v(E, \mathcal{G}),$$

and is independent of the rectangle R chosen. E is said to be a Jordan region if $\overline{\mathrm{Vol}}(E) = \underline{\mathrm{Vol}}(E)$. For a Jordan region E and a rectangle R containing E, the outer and inner volumes are equal and the volume of E is defined by

$$V(E) = \inf_{\mathcal{G} \text{ grid on } R} V(E, \mathcal{G}).$$

If $V(E) = 0$, E is said to have volume zero.

(c) [2] Suppose $E \subset \mathbb{R}^d$ is a Jordan region. Let $E \subseteq R$, a rectangle. We first observe that for any grid $\mathcal{G}$ on R,

$$V(E, \mathcal{G}) - v(E, \mathcal{G}) = V(\partial E, \mathcal{G}). \tag{1}$$

Suppose $V(\partial E) = 0$. Then $V(\partial E, \mathcal{G}) = V(E, \mathcal{G}) - v(E, \mathcal{G}) \geq \overline{\mathrm{Vol}}(E) - \underline{\mathrm{Vol}}(E)$. Taking the infimum, we get

$$0 = V(\partial E) \geq \overline{\mathrm{Vol}}(E) - \underline{\mathrm{Vol}}(E) \geq 0.$$

Hence E is a Jordan region. Conversely, suppose E is a Jordan region. Then $\overline{\mathrm{Vol}}(E) = \underline{\mathrm{Vol}}(E)$. For $\varepsilon > 0$, there exist grids $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\overline{\mathrm{Vol}}(E) + \varepsilon > V(E, \mathcal{H}_1)$ and $\underline{\mathrm{Vol}}(E) - \varepsilon <$

$v(E, \mathcal{H}_2)$. Taking $\mathcal{G}$ to be a refinement of $\mathcal{H}_1$ and $\mathcal{H}_2$, we get $\overline{\mathrm{Vol}}(E) + \varepsilon > V(E, \mathcal{G})$ and $\underline{\mathrm{Vol}}(E) - \varepsilon < v(E, \mathcal{G})$. Subtracting, we get

$$0 \leq V(\partial E, \mathcal{G}) = V(E, \mathcal{G}) - v(E, \mathcal{G}) < \overline{\mathrm{Vol}}(E) - \underline{\mathrm{Vol}}(E) + 2\varepsilon = 2\varepsilon.$$

Thus $V(\partial E) = 0$.

4. *Let U be an open subset of $\mathbb{R}^2$, let $K \subset U$ be a compact set. Suppose that $f : U \to \mathbb{R}$ is a continuously differentiable function. Let $E := \{(x, y) \in K | f(x, y) = 0\}$. Suppose df is never zero on E. Show that E is a set of area (that is, 2-volume) zero in $\mathbb{R}^2$.*

Solution:

Let $(a, b) \in E$. Then either $\frac{\partial f}{\partial x}(a, b) \neq 0$ or $\frac{\partial f}{\partial y}(a, b) \neq 0$. Assume without loss of generality that $\frac{\partial f}{\partial x}(a, b) \neq 0$. Then by the implicit function theorem, there exist open sets $U_{(a,b)}$ and $W_{(a,b)}$ such that $(a, b) \in U_{(a,b)}$, $b \in W_{(a,b)}$ and for every $y \in W_{(a,b)}$, there exists unique x such that $(x, y) \in U_{(a,b)}$ and $f(x, y) = 0$. We write $g_{(a,b)}(y) = x$, that is $f(g_{(a,b)}(y), y) = 0$. By compactness, there exists $n \in \mathbb{N}$ such that we have $E \subseteq \cup_{i=1}^{n} U_{(a_i,b_i)}$. Let $U_i = U_{(a_i,b_i)}, W_i = W_{(a_i,b_i)}$ and $g_{(a_i,b_i)} = g_i$. Then $E = \cup_{i=1}^{n} U_i \cap E$. Let $(x, y) \in U_i \cap E$. Then $f(x, y) = 0$ and $(x, y) \in U_i$. Observing the proof of the implicit function theorem, we see that this implies that $y \in W_i$. Hence $(x, y) = (g_i(y), y)$. That is,

$$U_i \cap E \subseteq \{(g_i(y), y) : y \in W_i\}.$$

For each i, $\mathcal{G}_i = \{(g_i(y), y) : y \in W_i\}$ is a Jordan region of Jordan volume 0 (as it is the graph of the function g_i). Hence $E \subseteq \cup_{i=1}^{n} \mathcal{G}_i$ is also a Jordan region with volume 0.

References

[1] Rudin, Walter."Principles of Mathematical Analysis (International Series in Pure and Applied Mathematics)" (1976).

[2] Wade, William. "An Introduction to Analysis" (2004).